Distributions of Functions of Normal Random Variables
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The Unit (or Standard) Normal

The unit or standard normal random variable $U$ is a normally distributed variable with mean zero and variance one, i.e. $U \sim \mathcal{N}(0, 1)$. Note that if $x \sim \mathcal{N}(\mu, \sigma^2)$ that

$$
\frac{x - \mu}{\sigma} \sim U \sim \mathcal{N}(0, 1)
$$

(1)

Thus to simulate a normal random variable with mean $\mu$ and variance $\sigma^2$, we can simply transform unit normals, as

$$
x \sim \mu + \sigma U \sim \mathcal{N}(\mu, \sigma^2)
$$

(2)

Consider $n$ independent random variables $x_i \sim \mathcal{N}(\mu, \sigma^2)$, then $\bar{x} \sim \mathcal{N}(\mu, \sigma^2/n)$, and this

$$
\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim U \sim \mathcal{N}(0, 1)
$$

(3)

**Example 1.** Let’s construct a 95% confidence interval for the mean $\mu$ for Equation (3). First, let’s use R to compute a value $U_{0.975}$ such that $\Pr(U \leq U_{0.975}) = 0.975$. In R, typing the command `qnorm(0.975)` returns 1.96. Likewise, `qnorm(0.025)` returns −1.96 and hence $\Pr(U \leq -1.96) = 0.025$. Hence,

$$
\Pr(-1.96 \leq U \leq 1.96) = 0.95
$$

Recalling Equation (3),

$$
\Pr(-1.96 \leq U \leq 1.96) = \Pr \left( -1.96 \leq \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq 1.96 \right)
$$

Rearranging gives

$$
\Pr \left( -1.96\sigma/\sqrt{n} \leq \bar{x} - \mu \leq 1.96\sigma/\sqrt{n} \right)
$$

or

$$
\Pr \left( -\bar{x} - 1.96\sigma/\sqrt{n} \leq -\mu \leq -\bar{x} + 1.96\sigma/\sqrt{n} \right)
$$
which can also be written as
\[
Pr \left( \bar{x} - 1.96\sigma / \sqrt{n} \leq \mu \leq \bar{x} + 1.96\sigma / \sqrt{n} \right) = 0.95
\]
giving a 95% confidence interval for the mean \( \mu \).

### Central and Noncentral \( \chi^2 \) Distributions

The \( \chi^2 \) distribution arises from sums of squared, normally distributed, random variables — if \( x_i \sim \mathcal{N}(0, 1) \), then \( u = \sum_{i=1}^{n} x_i^2 \sim \chi^2_n \), a central \( \chi^2 \) distribution with \( n \) degrees of freedom. It follows that the sum of two \( \chi^2 \) random variables is also \( \chi^2 \) distributed, so that if \( u \sim \chi^2_n \) and \( v \sim \chi^2_m \), then
\[
u + v \sim \chi^2_{(n+m)} \tag{4a}
\]
Two other useful results are that if \( x_i \sim \mathcal{N}(0, \sigma^2) \), then
\[
\sum_{i=1}^{n} x_i^2 \sim \sigma^2 \cdot \chi^2_n \tag{4b}
\]
and for \( \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \),
\[
\sum_{i=1}^{n} (x_i - \bar{x})^2 \sim \sigma^2 \cdot \chi^2_{(n-1)} \tag{4c}
\]
In this last case, subtraction of the mean causes the loss of one degree of freedom. Note that a special case of Equation (4c) is the sample estimate of the variance,
\[
\text{Var}(x) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2
\]
so that
\[
(n - 1)\text{Var}(x) \sim \sigma^2 \cdot \chi^2_n, \quad \text{implying} \quad \frac{(n - 1)\text{Var}(x)}{\sigma^2} \sim \chi^2_n \tag{4d}
\]

**Example 2.** We can use Equation (4d) to construct a confidence interval on the true variance \( \sigma^2 \) given the sample variance \( \text{Var}(x) \), provided the \( x_i \) are drawn from independent normals with the same mean and variance \( \sigma^2 \).
First, recall that the R command `qchisq(p, df)` returns a value $X$ such that $\Pr(\chi^2_{df} \leq X) = p$. Suppose sample size is $n = 20$. Since $qchisq(0.975, 19)$ returns a value of 32.85 and $qchisq(0.025, 19)$ returns 8.91, we have

$$\Pr(8.91 \leq \chi^2_{19} \leq 32.85) = 0.95$$

From Equation 4d,

$$\Pr(8.91 \leq \chi^2_{19} \leq 32.85) = \Pr \left( 8.91 \leq \frac{(n-1)\text{Var}(x)}{\sigma^2} \leq 32.85 \right)$$

Noting that for

$$\Pr(a \leq x \leq b) = \Pr \left( \frac{1}{a} \geq \frac{1}{x} \geq \frac{1}{b} \right)$$

we have

$$\Pr \left( 8.91 \leq \frac{19\text{Var}(x)}{\sigma^2} \leq 32.85 \right) = \Pr \left( \frac{1}{8.91} \geq \frac{\sigma^2}{19\text{Var}(x)} \geq \frac{1}{32.85} \right)$$

or

$$\Pr \left( \frac{19\text{Var}}{8.91} \geq \sigma^2 \geq \frac{19\text{Var}}{32.85} \right) = 0.95$$

or

$$\Pr \left( 2.13\text{Var} \geq \sigma^2 \geq 0.58\text{Var} \right) = 0.95$$

giving the 95% confidence interval on the variance as 0.58Var to 2.13Var.

A noncentral $\chi^2$ arises when the random variables being considered have nonzero means. In particular, if $x_i \sim N(\mu_i, 1)$, then $u = \sum_{i=1}^{n} x_i^2$ follows a noncentral $\chi^2$ distribution with $n$ degrees of freedom and noncentrality parameter

$$\lambda = \sum_{i=1}^{n} \mu_i^2 \quad \text{ (5a)}$$

and we write $u \sim \chi^2_{n, \lambda}$. As shown in Figure 1, increasing the noncentrality parameter $\lambda$ shifts the distribution to the right. This is also seen by considering the mean and variance of $u$,

$$E(u) = n + \lambda \quad \text{and} \quad \sigma^2(u) = 2(n + 2\lambda) \quad \text{ (5b)}$$
4 Functions of Normal Random Variables

It follows directly from the definition that sums of noncentral $\chi^2$ variables also follows a noncentral $\chi^2$ distribution, so that if $u \sim \chi^2_{n, \lambda_1}$ and $v \sim \chi^2_{m, \lambda_2}$, then

$$(u + v) \sim \chi^2_{(n+m), (\lambda_1 + \lambda_2)} \quad (5c)$$

Finally, Equations 4b,c can be generalized to noncentral $\chi^2$ random variables as follows. Suppose $x_i \sim N(\mu_i, \sigma^2)$, then

$$\sum_{i=1}^{n} x_i^2 \sim \sigma^2 \cdot \chi^2_{n, \lambda} \quad \text{where} \quad \lambda = \sum_{i=1}^{n} \frac{\mu_i^2}{\sigma^2} \quad (5d)$$

Turning the distribution of $\sum_{i=1}^{n} (x_i - \bar{x})^2$, defining

$$\lambda^* = \sum_{i=1}^{n} \frac{(\mu_i - \bar{\mu})^2}{\sigma^2}, \quad \text{where} \quad \bar{\mu} = \frac{1}{n} \sum_{i=1}^{n} \mu_i$$

then

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 \sim \sigma^2 \cdot \chi^2_{n, \lambda^*} \quad (5e)$$

Note that if all the $x_i$ have the same mean ($\mu_i = \mu = \bar{\mu}$), $\lambda^* = 0$ and the $\chi^2$ is central, while if there is some variance in the means of the $x_i$, then distribution is a noncentral $\chi^2$.

R provides commands for quantities of interest with noncentral $\chi^2$ distributions.

- **qchisq(p, df, ncp)** returns a value $X$ such that $P(\chi^2_{df, ncp} \leq X) = p$
Functions of Normal Random Variables

- `pchisq(X, df, ncp)` returns the probability that $\Pr(\chi^2_{df,ncp} \leq X)$
- leaving out the field for `ncp` returns these values for a central $\chi^2$.

**Student’s t Distribution**

If $x \sim N(\mu, \sigma^2)$, then for Equation 2, we have $(\bar{x} - \mu)/(\sigma/\sqrt{n}) \sim N(0, 1)$, which allows for both hypothesis testing and construction of confidence intervals when $\sigma^2$ is known. When the variance is unknown, the above test statistic replaces the true (but unknown) variance $\sigma^2$ with the sample variance $\text{Var}(x)$,

$$t = \frac{\bar{x} - \mu}{\sqrt{\text{Var}/n}} \quad (6)$$

Notice that

$$t = \left( \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \right) \left( \frac{1}{\sqrt{\text{Var}/\sigma^2}} \right) = \frac{U}{\sqrt{\chi^2_{n-1}/(n-1)}}$$

Thus, we define a $t$ distributed random variable with $\nu$ degrees of freedom by

$$t_\nu = \frac{U}{\sqrt{\chi^2_{\nu}}/\nu} \quad (7a)$$

A $t$ random variable has mean zero and variance

$$\sigma^2(t_\nu) = 1 + \frac{2}{\nu - 2} \quad \text{for} \quad \nu > 2 \quad (7b)$$

The coefficient of kurtosis is $k_4 = \frac{6}{\nu - 4}$, implying that the $t$ distribution has heavier tails than a normal.

The noncentral Student’s $t$ distribution is defined as follows: If $x \sim N(\mu_0, \sigma^2)$, but we assume the correct mean is $\mu$, then

$$t_{\nu, \lambda} = \frac{\bar{x} - \mu}{\sqrt{\text{Var}/n}}$$

is distributed as a noncentral $t$ random variable with $n - 1$ degrees of freedom and noncentrality parameter $\lambda = (\mu - \mu_0)/\sigma$.

**Central and Noncentral F Distributions**

The ratio of two $\chi^2$-distributed variables leads to the $F$ distribution. In particular, if $u \sim \chi^2_n$ and $v \sim \chi^2_m$, then the ratio of these two $\chi^2$ variables divided by their respective degrees of freedom follows a central $F$ distribution with numerator and denominator degrees of freedom $n$ and $m$ (respectively), i.e., $(u/n)/(v/m) \sim F_{n,m}$. Since

$$\lim_{m \to \infty} F_{n,m} \to \frac{\chi^2_n}{n}$$
the $F$ distribution can be approximated by a $\chi^2_n$ when the denominator degrees of freedom is large.

R provides commands for quantities of interest for $F$ distributions.

- \texttt{qf(p, df1, df2)} returns a value $X$ such that $\Pr(F_{df1,df2} \leq X) = p$
- \texttt{pf(X, df1, df2)} returns the probability that $\Pr(F_{df1,df2} \leq X)$

The \textbf{noncentral F distribution} results when the numerator $\chi^2$ variable is noncentral. If $u \sim \chi^2_{n,\lambda}$ and $v \sim \chi^2_{m}$, then $F = (u/n)/(v/m)$ follows a noncentral $F$ distributed with noncentrality parameter $\lambda$, and we write $F \sim F_{n,m,\lambda}$. As with the noncentral $\chi^2$, increasing $\lambda$ shifts the distribution further to the right. Again, this is seen in the mean and variance, with

\[
E(F) = \frac{n}{m-2} \left( 1 + \frac{2\lambda}{n} \right) \quad (A5.16a)
\]

\[
\sigma^2(F) = 2 \left( \frac{m}{n} \right)^2 \left[ \frac{(n+m)^2 + (n+2\lambda)(m-2)}{(m-2)^2(m-4)} \right] \quad (A5.16b)
\]

R provides commands for quantities of interest for noncentral $F$ distributions.
- \texttt{pf(X, df1, df2, ncp)} returns the probability that $\Pr(F_{df1,df2,ncp} \leq X)$
- the obvious command \texttt{qf(p, df1, df2, ncp)} does not work, as the same value is returned for all values of \texttt{ncp}. 