EEB 596z, Problem Set 5

Due 28 March 2002

1 : The **Weibul distribution** (after the Swedish physicist Waloddi Weibul, who proposed this distribution in 1939 for the breaking strenght of materials), has density function

$$p(x) = \lambda\, x^{\lambda-1} \exp(-x^{\lambda}) \quad \text{for} \quad x, \lambda > 0$$

[As an aside, note that the Weibul arises by assuming $y = x^{\lambda}$ follows an exponential distribution, $p(y) = \theta \exp(-\theta y)$].

Suppose we sample n values $(x_1, \cdots, x_n)$ independently from a Weibul with parameter λ.

(a) What is the resulting likelihood function $\ell(\lambda \,|\, x_1, \cdots, x_n)$, for λ?

$$\ell(\lambda \,|\, x_1, \cdots, x_n) = \lambda^n \left(\prod_{i=1}^{n} x_i^{\lambda-1} \right) \exp\left(- \sum_{i=1}^{n} x_i^{\lambda} \right)$$

(b) What is the resulting log-likelihood function?

$$\ln(\ell(\lambda \,|\, x_1, \cdots, x_n)) = n \ln \lambda + (\lambda - 1) \sum_{i=1}^{n} ln\, x_i - \sum_{i=1}^{n} x_i^{\lambda}$$

(c) What is the score function for λ?

$$S(\lambda) = \frac{\partial \ln(\ell(\lambda \,|\, x_1, \cdots, x_n))}{\partial \lambda} = \frac{\partial\; n \ln \lambda + (\lambda - 1) \sum_{i=1}^{n} ln\, x_i - \sum_{i=1}^{n} x_i^{\lambda}}{\partial \lambda}$$

Recalling that $d\, x^{\lambda} / d\, \lambda = x^{\lambda}\, \ln(x)$**, which follows since**

$$x^{\lambda} = e^{\ln(x^{\lambda})} = e^{\lambda \ln(x)}, \text{ so that } \frac{d], e^{\lambda \ln(x)}}{dx} = \ln(x) e^{\lambda \ln(x)} = \ln(x) x^{\lambda}$$

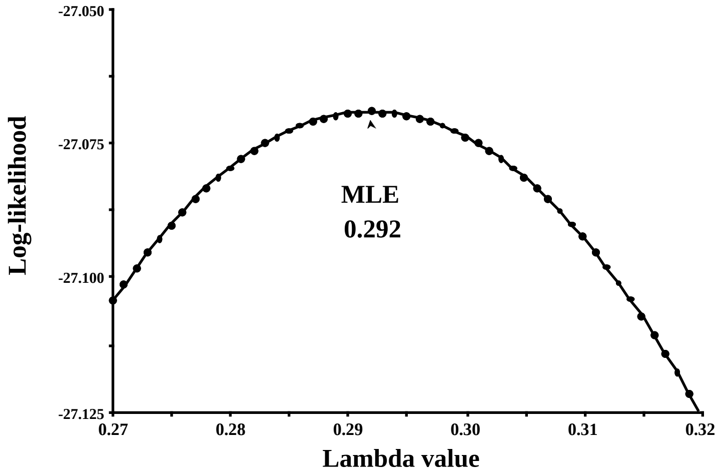
gives

$$= \frac{n}{\lambda} + \sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} x_i^{\lambda}\, \ln(x_i)$$

(d) What is the information (i.e., the derivate of the score function, which is the second derivative of the log-likelihood function)

$$\frac{\partial S(\lambda)}{\partial \lambda} = - \left(\frac{n}{\lambda^2} + \sum_{i=1}^{n} x_i^{\lambda}\, \ln(x_i)^2 \right)$$

(e) Suppose we observe 5 x values: 12, 15.5, 17.5, 21.3, and 13.5. Plot the log-likelihood as a function of λ (you can use Excel, or whatever program you wish).



(f) What is the MLE in this case? (You can use the previous graph)

MLE for λ is $\widehat{\lambda} = 0.292$

(g) Using your results from parts (d) and (f), what is the variance of your MLE estimator?

Recall from the notes (Equation A4.6), that

$$\sigma^2(\,\widehat{\lambda}\,) = -\left(\frac{\partial^2 \, \ln(\ell(\lambda \,|\, x_1, \cdots, x_n))}{\partial\, \lambda^2} \right)^{-1}$$

when the infromation (the second derivative) is evaluated at the MLE. Hence,

$$\begin{aligned}\sigma^2(\,\widehat{\lambda}\,) &= -\left(-\frac{n}{\lambda^2} - \sum_{i=1}^{n} x_i^{\lambda} \, \ln(x_i)^2 \right)^{-1} \\ &= -\left(-\frac{5}{0.292^2} + \sum_{i=1}^{n} x_i^{0.292} \, \ln(x_i)^2 \right)^{-1} \\ &= -(\,-144.357\,)^{-1} = 0.00692\end{aligned}$$

Hence, note than an approximate 95% confidence interval for λ is

$$\widehat{\lambda} + \pm 1.96\sqrt{0.00692} = 0.292 \pm 0.163$$

(h) What is the probability value for the likelihood-ratio test for this data that $\lambda = 1$? For $\lambda = 4$?

The likelihood ratio (LR) test for $\lambda = 1$ is

$$\textbf{LR} = 2\ln\left(\frac{0.292^5 \left(\prod_{i=1}^{5} x_i^{0.292-1}\right) \exp\left(-\sum_{i=1}^{5} x_i^{0.292}\right)}{1^5 \left(\prod_{i=1}^{n} x_i^{1-1}\right) \exp\left(-\sum_{i=1}^{5} x_i^{1}\right)} \right) = 105.461$$

LT is distributed as a χ^2 distribution with one degree of freedom. $\Pr(\chi_1^2 > 105.461) \simeq 0$.
The likelihood ratio (LR) test for $\lambda = 0.4$ gives LR = 1.66. $\Pr(\chi_1^2 > 1.66) = 0.198$.