

EEB 596z, Problem Set Four

1 : Suppose $x_1, x_2,$ and x_3 are multivariate normally distributed with means $\mu_1 = 1, \mu_2 = 0, \mu_3 = -2,$ and covariance structure

$$\sigma^2(x_1) = 3, \quad \sigma^2(x_2) = 4, \quad \sigma^2(x_3) = 6, \quad \sigma(x_1, x_2) = 1, \quad \sigma(x_1, x_3) = -1, \quad \sigma(x_2, x_3) = 2$$

Finally, define $y = x_1 - 3x_2 + 4x_3$ and $z = 3x_1 + 4x_2 - 7x_3.$

(a) Compute $\sigma^2(y).$

Since $y = \mathbf{a}^T \mathbf{x}$ where $\mathbf{a}^T = (1 \quad -3 \quad 4),$

$$\sigma^2(y) = \mathbf{a}^T \mathbf{V} \mathbf{a} = (1 \quad -3 \quad 4) \begin{pmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} = 73$$

(b) Compute $\sigma^2(z).$

Since $z = \mathbf{b}^T \mathbf{x}$ where $\mathbf{b}^T = (3 \quad 4 \quad -7),$

$$\sigma^2(y) = \mathbf{b}^T \mathbf{V} \mathbf{b} = (3 \quad 4 \quad -7) \begin{pmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} = 339$$

(c) Compute $\sigma(y, z), \rho(y, z)$

$$\sigma(y, z) = \mathbf{a}^T \mathbf{V} \mathbf{b} = (1 \quad -3 \quad 4) \begin{pmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} = -143$$

$$\rho(y, z) = \frac{\sigma(y, z)}{\sigma(x) \sigma(y)} = \frac{\mathbf{a}^T \mathbf{V} \mathbf{b}}{\sqrt{\mathbf{a}^T \mathbf{V} \mathbf{a} \cdot \mathbf{b}^T \mathbf{V} \mathbf{b}}} = \frac{-143}{\sqrt{73 \cdot 339}} = -0.909$$

(d) What is the distribution of x_1, x_2 given $x_3?$

Define

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = (x_3)$$

Thus

$$\boldsymbol{\mu}_{\mathbf{x}_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\mu}_{\mathbf{x}_2} = (-2), \quad \mathbf{V}_{\mathbf{x}_1 \mathbf{x}_1} = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{V}_{\mathbf{x}_1 \mathbf{x}_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{V}_{\mathbf{x}_2 \mathbf{x}_2} = (6)$$

$\mathbf{x}_1 | \mathbf{x}_2$ is MVN with mean vector

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{x}_1 | \mathbf{x}_2} &= \boldsymbol{\mu}_1 + \mathbf{V}_{\mathbf{x}_1 \mathbf{x}_2} \mathbf{V}_{\mathbf{x}_2 \mathbf{x}_2}^{-1} (\mathbf{x}_2 - \boldsymbol{\mu}_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} \frac{1}{6} (x_3 + 2) \\ &= \frac{1}{6} \begin{pmatrix} 4 - x_3 \\ 4 + 2x_3 \end{pmatrix} \end{aligned}$$

and covariance matrix

$$\begin{aligned} \mathbf{V}_{\mathbf{x}_1 | \mathbf{x}_2} &= \mathbf{V}_{\mathbf{x}_1 \mathbf{x}_1} - \mathbf{V}_{\mathbf{x}_1 \mathbf{x}_2} \mathbf{V}_{\mathbf{x}_2 \mathbf{x}_2}^{-1} \mathbf{V}_{\mathbf{x}_2 \mathbf{x}_1}^T \\ &= \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -1 \\ 2 \end{pmatrix} (-1 \quad 2) = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \\ &= \frac{1}{6} \begin{pmatrix} 17 & 8 \\ 8 & 20 \end{pmatrix} = \begin{pmatrix} 2.833 & 1.333 \\ 1.333 & 3.333 \end{pmatrix} \end{aligned}$$

(e) What is the regression of x_1 on x_2 and $x_3?$

$$\mathbf{x}_1 = \boldsymbol{\mu}_1 + \mathbf{V}_{\mathbf{x}_1\mathbf{x}_2} \mathbf{V}_{\mathbf{x}_2\mathbf{x}_2}^{-1} (\mathbf{x}_2 - \boldsymbol{\mu}_2)$$

where

$$\boldsymbol{\mu}_1 = (1), \quad \boldsymbol{\mu}_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad \mathbf{V}_{\mathbf{x}_2\mathbf{x}_2} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}, \quad \mathbf{V}_{\mathbf{x}_1\mathbf{x}_2} = (1 \quad -1), \quad \mathbf{V}_{\mathbf{x}_2\mathbf{x}_2}^{-1} = \begin{pmatrix} 0.3 & -0.1 \\ -0.1 & 0.2 \end{pmatrix}$$

Thus

$$x_1 = 1 + (1 \quad -1) \begin{pmatrix} 0.3 & -0.1 \\ -0.1 & 0.2 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 + 2 \end{pmatrix} = 1 + (0.4 \quad -0.3) \begin{pmatrix} x_2 \\ x_3 + 2 \end{pmatrix} = 0.4 + 0.4x_2 - 0.3x_3$$

(f) What is the conditional variance of x_1 given x_2 and x_3 ?

$$\begin{aligned} \mathbf{V}_{\mathbf{x}_1|\mathbf{x}_2} &= \mathbf{V}_{\mathbf{x}_1\mathbf{x}_1} - \mathbf{V}_{\mathbf{x}_1\mathbf{x}_2} \mathbf{V}_{\mathbf{x}_2\mathbf{x}_2}^{-1} \mathbf{V}_{\mathbf{x}_1\mathbf{x}_2}^T \\ &= 3 - (1 \quad -1) \begin{pmatrix} 0.3 & -0.1 \\ -0.1 & 0.2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3 - 0.7 = 2.3 \end{aligned}$$