EEB 581, Problem Set Nine Solutions

1 : Consider the following covariance matrix,

$$A = \begin{pmatrix} 40 & 30 & -30 \\ 30 & 40 & -20 \\ -30 & -20 & 40 \end{pmatrix}$$

(a) What is the trace of $\mathbf{A}$?.
trace($\mathbf{A}$) = 40+40+40 =120

(b) Use **R** to compute the determinant of $\mathbf{A}$.
det(A) = 12000

(c) Use **R** to solve for eigenvalues and the corresponding eigenvectors.
R returns

```
$values
[1] 93.58899 20.00000 6.41101
$vectors x
            [,1]           [,2]        [,3]
[1,]  -0.6207190    1.459452e-16   0.7840331
[2,]  -0.5543951    -7.071068e-01  -0.4389146
[3,]  0.5543951     -7.071068e-01  0.4389146
```

(d) Use **R** to show that the eigenvectors and their corresponding eigenvalues obtained in (c) satisfy the eigenvalue-eigenvector equation (i.e., that $\mathbf{A}\mathbf{e_i} = \lambda_i \mathbf{e_i}$).

(e) Using your results from (a) and (c), is trace ($\mathbf{A}$) = sum of the eigenvalues?
Yes, 93.58899 + 20.00000 + 6.41101 = 120

(f) Using your results from (a) and (c), is det($\mathbf{A}$) = product of the eigenvalues?
Yes, 93.58899 * 20.00000 * 6.41101 = 12000

(g) Diagonalize $\mathbf{A}$, i.e., write it as the matrix product $\mathbf{U}^T \Lambda \mathbf{U}$.

$$\mathbf{U} = \begin{pmatrix} -0.620 & 0 & 0.784 \\ -0.554 & -0.707 & -0.439 \\ 0.554 & -0.707 & 0.439 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} -93.59 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 6.41 \end{pmatrix}$$

Hence

$$\mathbf{A} = \begin{pmatrix} -0.620 & -0.554 & 0.554 \\ 0 & -0.707 & -0.707 \\ 0.784 & -0.439 & 0.439 \end{pmatrix} \begin{pmatrix} -93.59 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 6.41 \end{pmatrix} \begin{pmatrix} -0.620 & 0 & 0.784 \\ -0.554 & -0.707 & -0.439 \\ 0.554 & -0.707 & 0.439 \end{pmatrix}$$

(h) Use matrix multiplication in **R** to show indeed that $\mathbf{A} = \mathbf{U}^T \Lambda \mathbf{U}$.

```
U <- eigen(A)$vectors
Lambda <- diag(eigen(A)$values)
```

`U%*% Lambda%*% t(U)` returns A

(i) Use matrix multiplication in **R** to show that $\mathbf{U}^T\mathbf{U} = \mathbf{U}\mathbf{U}^T = \mathbf{I}$.
`t(U)%*%U` and `U%*%t(U)` both return $\mathbf{I}$.

(j) What are the eigenvalues for $\mathbf{A}^3$? $\mathbf{A}^{-1}$? $\mathbf{A}^{1/2}$? $\mathbf{A}^{-1/2}$?

```
eval <-eigen(A)$values
```

For $\mathbf{A}^3$,**eval∧3** returns $\lambda^3 = 819736.5155, \quad 8000.0000, \quad 263.4992.$
For $\mathbf{A}^{-1}$, **eval∧(-1)** returns $\lambda^{-1} = 0.01068502, \quad 0.05000000, \quad 0.15598166.$
For $\mathbf{A}^{1/2}$, **eval∧(1/2)** returns $\lambda^{1/2} = 0.674140, \quad 4.472136, \quad 2.531997.$
For $\mathbf{A}^{-1/2}$, **eval∧(-1/2)** returns $\lambda^{-1/2} = 0.1033684, \quad 0.2236068, \quad 0.3949451.$

(k) Use diagonalization to compute these four matrices.

U%*% Lambda∧3 %*% t(U) returns $\mathbf{A}^3$

U%*% diag(eigen(A)$values∧(-1))%*% t(U) returns $\mathbf{A}^{-1}$ (Need to be careful here, as if the matrix has any zero's, 1/0 returns error message. Hence, took the inverse of each element in the eigenvalue vector and then used this for creation of the diagonal matrix.

U%*% Lambda∧(1/2)%*% t(U) returns $\mathbf{A}^{1/2}$

U%*% diag(eigen(A)$values∧(-1/2))%*% t(U) returns $\mathbf{A}^{-1/2}$

(l) Use **R** to show that $\mathbf{A}^{1/2}\mathbf{A}^{1/2}$=$\mathbf{A}$, with $\mathbf{A}$ obtained as in (j).

sqA <- U%*% Lambda∧(1/2)%*% t(U)

sqA%*%sqA does indeed return $\mathbf{A}$

(m) For this covariance matrix (which is the vector $\mathbf{x}^T = (x_1, x_2, x_3)$, which linear combination of the x_i corresponds to PC 1? How much of the total variation does PC 1 explain?

PC 1 = -0.620$\cdot x_1$ -0.554 $\cdot x_2$ +0.554$\cdot x_3$. The precent of total variation for PC 1 is 93.59/120 = 0.78, for 78%.

(n) What linear combination gives PC 2? How much of the total variation does PC 2 explain?

PC 2 = -(x_2 + $\cdot x_3$). The precent of total variation for PC 1 is 20/120 = 0.17, for 17%.