EEB 581, Problem Set Nine, Due 4 April 2006

Some helpful R commands before starting!
- \texttt{t(A)} computes the transpose of A.
- \texttt{det(A)} computes the determinant of A.
- \texttt{diag(c)} takes an \( n \)-dimensional column vector \( c \) creates a diagonal matrix whose \( i \)-th diagonal element is the \( i \)-th element of \( c \).
- \texttt{eigen(A)} computes the eigenvalues (as a row vector, with the \( i \)-th element the \( i \)-th eigenvalue) and eigenvectors (as a matrix, whose \( i \)-th column is the \( i \)-th eigenvector) of \( A \).
- \texttt{eigen(A)$values} directly returns the row vector of eigenvalues, so that \texttt{eval<- eigen(A)$values} stores this vector for future use.
- Note that if \( b \) is a vector, then the R command \( b^5 \) returns a vector where every element has been raised to the 5-th power.
- \texttt{eigen(A)$vectors} directly returns the matrix of eigenvectors.

1: Consider the following covariance matrix,

\[
A = \begin{pmatrix}
40 & 30 & -30 \\
30 & 40 & -20 \\
-30 & -20 & 40
\end{pmatrix}
\]

(a) What is the trace of \( A \)?
(b) Use R to compute the determinant of \( A \).
(c) Use R to solve for eigenvalues and the corresponding eigenvectors.
(d) Use R to show that the eigenvectors and their corresponding eigenvalues obtained in (c) satisfy the eigenvalue-eigenvector equation (i.e., that \( A e_i = \lambda_i e_i \)).
(e) Using your results from (a) and (c), is trace (\( A \)) = sum of the eigenvalues?
(f) Using your results from (a) and (c), is \( \text{det}(A) = \text{product of the eigenvalues} \)?
(g) Diagonalize \( A \), i.e., write it as the matrix product \( \mathbf{U} \Lambda \mathbf{U}^T \).
(h) Use matrix multiplication in R to show indeed that \( A = \mathbf{U} \Lambda \mathbf{U}^T \).
(i) Use matrix multiplication in R to show that \( \mathbf{U}^T \mathbf{U} = \mathbf{U} \mathbf{U}^T = \mathbf{I} \).
(j) What are the eigenvalues for \( A^3 \)\? \( A^{-1} \)\? \( A^{1/2} \)\? \( A^{-1/2} \)\?
(k) Use diagonalization to compute these four matrices.
(l) Use R to show that \( A^{1/2} A^{1/2} = A \), with \( A \) obtained as in (j).
(m) For this covariance matrix (which is the vector \( x^T = (x_1, x_2, x_3) \), which linear combination of the \( x_i \) corresponds to PC 1\? How much of the total variation does PC 1 explain?
(n) What linear combination gives PC 2\? How much of the total variation does PC 2 explain?