EEB 581, Problem Set Six

Solutions

1 : Consider the following linear models

(a) For $y = \mu + \beta_1 x_1 + \beta_2 x_2 + e$, what is the expected change in y given a one unit change in x_1?

β_1

(b) For $y = \mu + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 \cdot x_2 + e$, what is the expected change in y given a one unit change in x_1?

$\beta_1 + \beta_3 x_2$

(c) For the model in (b), what is the expected change given a one unit change in x_2?

$\beta_2 + \beta_3 x_1$

2 : Consider quadratic regression forced through the origin, $y_i = \beta_1 x_i + \beta_2 x_i^2 + e$.

(a) For n observations, write this in matrix form.

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}, \quad \text{where} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x_1 & x_1^2 \\ \vdots & \vdots \\ x_n & x_n^2 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

(b) What is the OLS estimator for β_1 and β_2.

$$\widehat{\boldsymbol{\beta}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1} \mathbf{X}^T\mathbf{y}$$

$$\mathbf{X}^T\mathbf{X} = \begin{pmatrix} \sum x_i^2 & \sum x_i^3 \\ \sum x_i^3 & \sum x_i^4 \end{pmatrix}, \quad \left(\mathbf{X}^T\mathbf{X}\right)^{-1} = \frac{1}{\left(\sum x_i^2\right)\left(\sum x_i^4\right) - \left(\sum x_i^3\right)^2} \begin{pmatrix} \sum x_i^4 & -\sum x_i^3 \\ -\sum x_i^3 & \sum x_i^2 \end{pmatrix}$$

$$\mathbf{X}^T\mathbf{y} = \begin{pmatrix} \sum x_i y_i \\ \sum x_i^2 y_i \end{pmatrix}$$

putting these together,

$$\widehat{\boldsymbol{\beta}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1} \mathbf{X}^T\mathbf{y} = \frac{1}{\left(\sum x_i^2\right)\left(\sum x_i^4\right) - \left(\sum x_i^3\right)^2} \begin{pmatrix} \sum x_i^4 & -\sum x_i^3 \\ -\sum x_i^3 & \sum x_i^2 \end{pmatrix} \begin{pmatrix} \sum x_i y_i \\ \sum x_i^2 y_i \end{pmatrix}$$

$$= \frac{1}{\left(\sum x_i^2\right)\left(\sum x_i^4\right) - \left(\sum x_i^3\right)^2} \begin{pmatrix} \left(\sum x_i^4\right)\left(\sum x_i y_i\right) - \left(\sum x_i^3\right)\left(\sum x_i^2 y_i\right) \\ -\left(\sum x_i^3\right)\left(\sum x_i y_i\right) + \left(\sum x_i^4\right)\left(\sum x_i^2 y_i\right) \end{pmatrix}$$

(c) What is the variance-covariance matrix for these estimates?

$$\left(\mathbf{X}^T\mathbf{X}\right)^{-1} = \frac{\sigma_e^2}{\left(\sum x_i^2\right)\left(\sum x_i^4\right) - \left(\sum x_i^3\right)^2} \begin{pmatrix} \sum x_i^4 & -\sum x_i^3 \\ -\sum x_i^3 & \sum x_i^2 \end{pmatrix}, \quad \sigma_e^2 = \frac{\sum (y_i - \widehat{y_i})^2}{n-2}$$

4 : Suppose you have the following data for 50 observations

$$\sum_i x_i^2 = 300, \quad \sum_i x_i^3 = 100, \quad \sum_i x_i^4 = 12000, \quad \sum_i x_i y_i = -200, \quad \sum_i x_i^2 y_i = 600$$

(a) Compute the OLS estimate of β_1 and β_2.

```
> XY <- matrix(c(-200,600),nrow=2)
> X <- matrix(c(300,100,100,12000),nrow=2)
> solve(X)%*%XY
```

R returns

```
                [,1]
  [1,] -0.68523677
  [2,]  0.05571031
```

(b) Suppose $\sum(y_i - \widehat{y_i})^2 = 400$. **Estimate** σ_e^2

$$\sigma_e^2 = 400/(50-2) = 8.33$$

(c) Compute $\sigma^2(\widehat{\beta_1})$.

$$\mathbf{cov}(\boldsymbol{\beta}) = \sigma_e^2 \left(\mathbf{X}^T \mathbf{X}\right)^{-1}$$

In R

```
> 8.33* solve(X)
```

R returns

```
                 [,1]           [,2]
  [1,]  0.0278440111   -0.0002320334
  [2,] -0.0002320334    0.0006961003
```

Hence $\sigma^2(\widehat{\beta_1}) = 0.0278440111$

(d) Compute $\sigma^2(\widehat{\beta_2})$.

$$\sigma^2(\widehat{\beta_2}) = 0.000696$$

(e) Compute $\sigma(\widehat{\beta_1}, \widehat{\beta_2})$

$$\sigma(\widehat{\beta_1}, \widehat{\beta_2}) = -0.000232$$