EEB 581, Problem Set Five

1 : Consider the matrix $\mathbf{A}$ and vector $\mathbf{b}$

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

By hand, compute the following (its OK to check the results in R):

(a) $\mathbf{Ab} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 3 \cdot 6 \\ 3 \cdot 5 + 2 \cdot 6 \end{pmatrix} = \begin{pmatrix} 23 \\ 27 \end{pmatrix}$

(b) $\mathbf{b}^T\mathbf{A} = \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 5 \cdot 1 + 6 \cdot 3 & 5 \cdot 3 + 6 \cdot 2 \end{pmatrix} = \begin{pmatrix} 23 & 27 \end{pmatrix}$

(c) $\mathbf{b}^T\mathbf{b} = \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = 5^2 + 6^2 = 61$

(d) $\mathbf{bb}^T = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \begin{pmatrix} 5 & 6 \end{pmatrix} = \begin{pmatrix} 5 \cdot 5 & 5 \cdot 6 \\ 6 \cdot 5 & 6 \cdot 6 \end{pmatrix} = \begin{pmatrix} 25 & 30 \\ 30 & 35 \end{pmatrix}$

(e) $\det(\mathbf{A}) = 1 \cdot 2 - 3 \cdot 3 = -7$

(f) $\mathbf{A}^{-1} = \frac{1}{-7} \begin{pmatrix} 2 & -3 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2/7 & 3/7 \\ 3/7 & -1/7 \end{pmatrix}$

2 : Consider the following set of equations

$$\begin{aligned} 8x_1 + 13x_2 - 4x_3 + x_4 &= 9 \\ -4x_1 + x_2 + 5x_3 - 3x_4 &= 5 \\ 7x_1 + 9x_2 + 2x_3 + 7x_4 &= -4 \\ 3x_1 + 4x_2 + 6x_3 + 2x_4 &= 12 \end{aligned}$$

(a) Express this system of equations in matrix form

$$\begin{pmatrix} 8 & 13 & -4 & 4 \\ -4 & 1 & 5 & -3 \\ 7 & 9 & 2 & 7 \\ 3 & 4 & 6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ -4 \\ 14 \end{pmatrix}$$

(b) Solve for the vector of the unknown x_i (ok to use **R** here!).
in **R**, use the **solve** command:

```
>X <- matrix(c(8,-4,7,3,13,1,9,4,-4,5,2,6,1,-3,7,2),nrow=4)
> y <- matrix(c(9,5,-4,12),nrow=4)
> solve(X)%*%y
```

R returns

```
      [,1]
[1,]  3.9228457
[2,]  -0.8261523
[3,]  1.9158317
[4,]  -3.9794589
```

3 : Suppose the vector x_1, x_2, x_3 has covariance matrix

$$\mathbf{V} = \begin{pmatrix} 12 & 1 & -3 \\ 1 & 10 & -2 \\ -3 & -2 & 50 \end{pmatrix}$$

Consider two new random variables,

$$y = 3x_1 + 2x_2 - 5x_3, \qquad z = x_1 - 12x_2 + 2x_3$$

(a) Compute $\sigma^2(y)$. (OK to use **R** throughout)

$$\sigma^2(y) = \sigma^2(\mathbf{a}^T\mathbf{x}) = \mathbf{a}^T\mathbf{V}\mathbf{a} = 1540, \quad \text{where} \quad \mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$$

(b) Compute $\sigma^2(z)$.

$$\sigma^2(y) = \sigma^2(\mathbf{b}^T\mathbf{x}) = \mathbf{b}^T\mathbf{V}\mathbf{b} = 1712, \quad \text{where} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -12 \\ 2 \end{pmatrix}$$

(c) Compute the correlation between y and y — recall $\rho(y,z) = \sigma(y,z)/\sqrt{\sigma^2(y)\cdot\sigma^2(z)}$

$$\sigma(y,z) = \mathbf{a}^T\mathbf{V}\mathbf{b} = -869, \quad \rho(y,z) = \frac{-869}{\sqrt{1540\cdot 1712}} = -0.535$$

4 : Suppose x_1, x_2, and x_3 are multivariate normally distributed with means $\mu_1 = 1$, $\mu_2 = 0$, $\mu_3 = -2$, and covariance structure

$$\sigma^2(x_1) = 3, \quad \sigma^2(x_2) = 4, \quad \sigma^2(x_3) = 6, \quad \sigma(x_1,x_2) = 1, \quad \sigma(x_1,x_3) = -1, \quad \sigma(x_2,x_3) = 2$$

Finally, define $y = x_1 - 3x_2 + 4x_3$ and $z = 3x_1 + 4x_2 - 7x_3$.

(a) Compute $\sigma^2(y)$.

Since $y = \mathbf{a}^T\mathbf{x}$ where $\mathbf{a}^T = (1 \quad -3 \quad 4)$,

$$\sigma^2(y) = \mathbf{a}^T\mathbf{V}\mathbf{a} = (1 \quad -3 \quad 4)\begin{pmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} = 73$$

(b) Compute $\sigma^2(z)$.

Since $z = \mathbf{b}^T\mathbf{x}$ where $\mathbf{b}^T = (3 \quad 4 \quad -7)$,

$$\sigma^2(y) = \mathbf{b}^T\mathbf{V}\mathbf{b} = (3 \quad 4 \quad -7)\begin{pmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} = 339$$

(c) Compute $\sigma(y,z)$, $\rho(y,z)$

$$\sigma(y,z) = \mathbf{a}^T\mathbf{V}\mathbf{b} = (1 \quad -3 \quad 4)\begin{pmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} = -143$$

$$\rho(y,z) = \frac{\sigma(y,z)}{\sigma(x)\,\sigma(y)} = \frac{\mathbf{a}^T\mathbf{V}\mathbf{b}}{\sqrt{\mathbf{a}^T\mathbf{V}\mathbf{a}\cdot\mathbf{b}^T\mathbf{V}\mathbf{b}}} = \frac{-143}{\sqrt{73\cdot 339}} = -0.909$$

(d) What is the distribution of x_1, x_2 given x_3?

Define

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \textbf{and} \quad \mathbf{x}_2 = (\, x_3 \,)$$

Thus

$$\boldsymbol{\mu}_{\mathbf{X}_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\mu}_{\mathbf{X}_2} = (\, -2 \,), \quad \mathbf{V}_{\mathbf{X}_1\mathbf{X}_1} = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{V}_{\mathbf{X}_1\mathbf{X}_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{V}_{\mathbf{X}_2\mathbf{X}_2} = (\, 6 \,)$$

$\mathbf{x_1}|\mathbf{x_2}$ **is MVN with mean vector**

$$\begin{aligned}
\boldsymbol{\mu}_{\mathbf{X_1}|\mathbf{X_2}} &= \boldsymbol{\mu_1} + \mathbf{V}_{\mathbf{X_1X_2}}\mathbf{V}^{-1}_{\mathbf{X_2X_2}}(\mathbf{x_2} - \boldsymbol{\mu_2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} \frac{1}{6}(x_3 + 2) \\
&= \frac{1}{6}\begin{pmatrix} 4 - x_3 \\ 4 + 2x_3 \end{pmatrix}
\end{aligned}$$

and covariance matrix

$$\begin{aligned}
\mathbf{V}_{\mathbf{X_1}|\mathbf{X_2}} &= \mathbf{V}_{\mathbf{X_1X_1}} - \mathbf{V}_{\mathbf{X_1X_2}}\mathbf{V}^{-1}_{\mathbf{X_2X_2}}\mathbf{V}^{T}_{\mathbf{X_1X_2}} \\
&= \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} -1 \\ 2 \end{pmatrix}(\, -1 \quad 2 \,) = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} 17 & 8 \\ 8 & 20 \end{pmatrix} = \begin{pmatrix} 2.833 & 1.333 \\ 1.333 & 3.333 \end{pmatrix}
\end{aligned}$$

(e) What is the regression of x_1 on x_2 and x_3?

$$\mathbf{x_1} = \boldsymbol{\mu_1} + \mathbf{V}_{\mathbf{X_1X_2}}\mathbf{V}^{-1}_{\mathbf{X_2X_2}}(\mathbf{x_2} - \boldsymbol{\mu_2})$$

where

$$\boldsymbol{\mu_1} = (\, 1 \,), \; \boldsymbol{\mu_2} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \; \mathbf{V}_{\mathbf{X_2X_2}} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}, \; \mathbf{V}_{\mathbf{X_1X_2}} = (\, 1 \quad -1 \,), \quad \mathbf{V}^{-1}_{\mathbf{X_2X_2}} = \begin{pmatrix} 0.3 & -0.1 \\ -0.1 & 0.2 \end{pmatrix}$$

Thus

$$x_1 = 1 + (\, 1 \quad -1 \,)\begin{pmatrix} 0.3 & -0.1 \\ -0.1 & 0.2 \end{pmatrix}\begin{pmatrix} x2 \\ x_3 + 2 \end{pmatrix} = 1 + (\, 0.4 \quad -0.3 \,)\begin{pmatrix} x2 \\ x_3 + 2 \end{pmatrix} = 0.4 + 0.4x_2 - 0.3x_3$$

(f) What is the conditional variance of x_1 given x_2 and x_3?

$$\begin{aligned}
\mathbf{V}_{\mathbf{X_1}|\mathbf{X_2}} &= \mathbf{V}_{\mathbf{X_1X_1}} - \mathbf{V}_{\mathbf{X_1X_2}}\mathbf{V}^{-1}_{\mathbf{X_2X_2}}\mathbf{V}^{T}_{\mathbf{X_1X_2}} \\
&= 3 - (\, 1 \quad -1 \,)\begin{pmatrix} 0.3 & -0.1 \\ -0.1 & 0.2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3 - 0.7 = 2.3
\end{aligned}$$