Consider the following data, assuming the $x_i$ are iid normals.

$$\bar{x} = 10, \quad \sum_{i=1}^{15} (x_i - \bar{x})^2 = 100, \quad n = 15$$

1: Use R to find the following:

(i) $X$ such that $\Pr(U \leq X) = 0.25$

- $\text{qnorm}(0.25)$ returns $-0.6744898$

(ii) $X$ such that $\Pr(U \leq X) = 0.75$

- $\text{qnorm}(0.75)$ returns $0.6744898$

(iii) $X$ such that $\Pr(U \leq X) = 0.005$

- $\text{qnorm}(0.005)$ returns $-2.575829$

(iv) $X$ such that $\Pr(U \leq X) = 0.995$

- $\text{qnorm}(0.995)$ returns $2.575829$

(v) Redo i - iv replacing the Unit normal $U$ with $t_{14}$, a $t$-distribution with 14 df.

- $\text{qt}(0.25, 14)$ returns $-0.6924171$
- $\text{qt}(0.75, 14)$ returns $0.6924171$
- $\text{qt}(0.005, 14)$ returns $-2.976843$
- $\text{qt}(0.995, 14)$ returns $2.976843$

2: Suppose we know that the true variance for $x$ is $\sigma_x^2 = 4$. Compute 50% and 99% confidence intervals for the true mean $\mu_x$. (Hint: Results from Problem 1 many be useful!)

- $\Pr(-0.674 \leq U \leq 0.674) = \Pr(-0.674 \leq \frac{10 - \mu}{\frac{1}{\sqrt{15}}} \leq 0.674) = 0.5$

Giving

$$\Pr(10 - 0.674 \cdot 2/\sqrt{15} \leq \mu \leq 10 + 0.674 \cdot 2/\sqrt{15}) = \Pr(9.65298 \leq \mu \leq 10.34702) = 0.5$$

(note that you can do these calculation in R, e.g., typing $10-0.672*2/sqrt(15)$ in the command line returns $9.65298$). Likewise,

$$\Pr(10 - 2.576 \cdot 2/\sqrt{15} \leq \mu \leq 10 + 2.576 \cdot 2/\sqrt{15}) = \Pr(8.66976 \leq \mu \leq 11.33024) = 0.99$$

3: Redo Problem 2, but now assume that the variance is estimated (and hence a $t$-distribution is needed).

**Here $Var(x) = 100/14 = 7.14$, the the resulting $t$ distribution has 14 degrees of freedom**

- $\Pr(-0.692 \leq t_{14} \leq 0.692) = \Pr(-0.692 \leq \frac{10 - \mu}{\sqrt{7.14/15}} \leq 0.692) = 0.5$

- $\Pr(10 - 0.692 \cdot \sqrt{7.14/15} \leq \mu \leq 10 + 0.692 \cdot \sqrt{7.14/15}) = \Pr(9.52257 \leq \mu \leq 10.47743) = 0.5$

Likewise

- $\Pr(10 - 2.977 \cdot \sqrt{7.14/15} \leq \mu \leq 10 + 2.977 \cdot \sqrt{7.14/15}) = \Pr(7.946086 \leq \mu \leq 12.05391) = 0.99$

4: Assuming the appropriate normality assumptions, compute the 95% confidence interval for $\sigma_x^2$. 
Recall (from the class notes) that \((n - 1)Var(x) \sim \sigma^2 \cdot \chi_{n-1}^2\). We thus need to find \(X_l\) and \(X_u\) such that 
\[
\Pr(\chi_{14}^2 \leq X_l) = 0.025 \quad \text{and} \quad \Pr(\chi_{14}^2 \leq X_u) = 0.975.
\]
Using R, `qchisq(0.025, 14)` returns 5.628726, while `qchisq(0.975, 14)` returns 26.11895. Hence
\[
\Pr(5.628726 \leq \chi_{14}^2 \leq 26.11895) = 0.95
\]
Thus
\[
\Pr(5.628726 \leq \frac{(n - 1)Var(x)}{\sigma^2} \leq 26.11895) = 0.95
\]
or, since \((n - 1)Var(x) = \sum_{i=1}^{15} (x_i - \bar{x})^2 = 100\),
\[
\Pr(5.628726 \leq \frac{100}{\sigma^2} \leq 26.11895) = 0.95
\]
or
\[
\Pr(1/5.628726 \geq \frac{\sigma^2}{100} \geq 1/26.11895) = 0.95
\]
giving
\[
\Pr(100/5.628726 \geq \sigma^2 \geq 100/26.11895) = \Pr(17.76601 \geq \sigma^2 \geq 3.828638) = 0.95
\]
5: Under the null hypothesis that the true variance is 20, how likely are we to observe a value as small as the observed sample variance (which is \(100/14 = 7.14\))? Hint: Express this as a \(\chi^2\) probability, as we want to find \(\Pr(Var(x) \leq 7.14)\).

Since
\[
\frac{(n - 1)/\sigma^2}{\cdot Var(x)} \sim \chi_{(n-1)}^2
\]
under the null we have that
\[
(14/20)Var(x) = 0.7 \cdot Var(x) \sim \chi_{(14)}^2
\]
Hence under the null hypothesis that \(\sigma^2 = 20\), while we observe \(Var(x) = 100/14 = 7.14\)
\[
\Pr(Var(x) \leq 7.14 \mid \sigma^2 = 20)
\]
\[
= \Pr(0.7 \cdot Var(x) \leq 0.7 \cdot 7.14) = \Pr(\chi_{14}^2 \leq 0.7 \cdot 7.14) = \Pr(\chi_{14}^2 \leq 4.998)
\]
Using R, `pchisq(4.998, 14)` returns 0.01415950, giving the probability of this as 1.4%.