EEB 581, Problem Set Two
Solutions

Consider the following data, assuming the x_i are iid normals.

$$\overline{x} = 10, \qquad \sum_{i=1}^{15} (x_i - \overline{x})^2 = 100, \qquad n = 15$$

1: Use **R** to find the following:

(i) X such that $\Pr(U \leq X) = 0.25$

qnorm(0.25) returns `-0.6744898`

(ii) X such that $\Pr(U \leq X) = 0.75$

qnorm(0.75) returns `0.6744898`

(iii) X such that $\Pr(U \leq X) = 0.005$

qnorm(0.005) returns `-2.575829`

(iv) X such that $\Pr(U \leq X) = 0.995$

qnorm(0.995) returns `2.575829`

(v) Redo i - iv replacing the Unit normal U with t_{14} ,a t-distribution with 14 df.

qt(0.25,14) returns `-0.6924171`
qt(0.75,14) returns `0.6924171`
qt(0.005,14) returns `-2.976843`
qt(0.995,14) returns `2.976843`

2: Suppose we know that the true variance for x is $\sigma_x^2 = 4$. Compute 50% and 99% confidence intervals for the true mean μ_x. (Hint: Results from Problem 1 many be useful!)

$$\Pr(-0.674 \leq U \leq 0.674) = \Pr(-0.674 \leq \frac{10-\mu}{2/\sqrt{15}} \leq 0.674) = 0.5$$

Giving

$$\Pr(10 - 0.674 \cdot 2/\sqrt{15} \leq \mu \leq 10 + 0.674 \cdot 2/\sqrt{15}) = \Pr(9.65298 \leq \mu \leq 10.34702) = 0.5$$

(note that you can do these calculation in R, e.g., typing `10-0.672*2/sqrt(15)` in the command line returns `9.65298`). Likewise,

$$\Pr(10 - 2.576 \cdot 2/\sqrt{15} \leq \mu \leq 10 + 2.576 \cdot 2/\sqrt{15}) = \Pr(8.66976 \leq \mu \leq 11.33024) = 0.99$$

3: Redo Problem 2, but now assume that the variance is estimated (and hence a t-distribution is needed).

Here $Var(x) = 100/14 = 7.14$, the the resulting t distribution has 14 degrees of freedom

$$\Pr(-0.692 \leq t_{14} \leq 0.692) = \Pr(-0.692 \leq \frac{10-\mu}{\sqrt{7.14/15}} \leq 0.692) = 0.5$$

$$\Pr(10 - 0.692 \cdot \sqrt{7.14/15} \leq \mu \leq 10 + 0.692 \cdot \sqrt{7.14/15}) = \Pr(9.52257 \leq \mu \leq 10.47743) = 0.5$$

Likewise

$$\Pr(10 - 2.977 \cdot \sqrt{7.14/15} \leq \mu \leq 10 + 2.977 \cdot \sqrt{7.14/15}) = \Pr(7.946086 \leq \mu \leq 12.05391) = 0.99$$

4: Assuming the appropriate normality assumptions, compute the 95% confidence interval for σ_x^2.

Recall (from the class notes) that $(n-1)Var(x) \sim \sigma^2 \cdot \chi^2_{n-1}$. We thus need to find X_l and X_u such that Pr($\chi^2_{14} \le X_l) = 0.025$ and Pr($\chi^2_{14} \le X_u) = 0.975$. Using `R`, `qchisq(0.025, 14)` returns `5.628726`, while `qchisq(0.025, 14)` returns `26.11895`. Hence

$$\Pr(5.628726 \le \chi^2_{14} \le 26.11895) = 0.95$$

Thus

$$\Pr(5.628726 \le \frac{(n-1)Var(x)}{\sigma^2} \le 26.11895) = 0.95$$

or, since $(n-1)Var(x) = \sum_{i=1}^{15} (x_i - \overline{x})^2 = 100$,

$$\Pr(5.628726 \le \frac{100}{\sigma^2} \le 26.11895) = 0.95$$

or

$$\Pr(1/5.628726 \ge \frac{\sigma^2}{100} \ge 1/26.11895) = 0.95$$

giving

$$\Pr(100/5.628726 \ge \sigma^2 \ge 100/26.11895) = \Pr(17.76601 \ge \sigma^2 \ge 3.828638) = 0.95$$

5: Under the null hypothesis that the true variance is 20, how likely are we to observe a value as small as the observed sample variance (which is 100/14 = 7.14)? Hint: Express this as a χ^2 probability, as we want to find $\Pr(\text{Var}(x) \le 7.14)$.

Since

$$[\,(n-1)/\sigma_2^2\,] \cdot \textbf{Var}(x) \sim \chi^2_{(n-1)}$$

under the null we have that

$$(14/20)\textbf{Var}(x) = 0.7 \cdot \textbf{Var}(x) \sim \chi^2_{(14)}$$

Hence under the null hypothesis that $\sigma^2 = 20$, while we observe Var$(x) = 100/14 = 7.14$

$$\Pr(\textbf{Var}(x) \le 7.14 \,|\, \sigma^2 = 20)$$

$$= \Pr(0.7 \cdot \textbf{Var}(x) \le 0.7 \cdot 7.14) = \Pr(\chi^2_{14} \le 0.7 \cdot 7.14) = \Pr(\chi^2_{14} \le 4.998)$$

Using `R`, `pchisq(4.998, 14)` returns `0.01415950`, giving the probability of this as 1.4%.