EEB 581, Problem Set Seven

Solutions

1 : The **Weibul distribution** (after the Swedish physicist Waloddi Weibul, who proposed the distribution in 1939 for the breaking strenght of materials), has density function

$$p(x) = \lambda x^{\lambda-1} \exp(-x^\lambda) \quad \text{for } x, \lambda > 0$$

[As an aside, note that the Weibul arises by assuming $y = x^\lambda$ follows an exponential distribution, $p(y) = \theta \exp(-\theta y)$].

(a) What is the resulting likelihood function $\ell(\lambda \,|\, x_1, \cdots, x_n)$, for λ?

$$\ell(\lambda \,|\, x_1, \cdots, x_n) = \lambda^n \left(\prod_{i=1}^n x_i^{\lambda-1} \right) \exp\left(-\sum_{i=1}^n x_i^\lambda \right)$$

(b) What is the resulting log-likelihood function?

$$\ln(\ell(\lambda \,|\, x_1, \cdots, x_n)) = n \ln \lambda + (\lambda - 1) \sum_{i=1}^n \ln x_i - \sum_{i=1}^n x_i^\lambda$$

(c) What is the score function? (Helpful hint, $dx^\lambda / d\lambda = x^\lambda \ln(x)$).

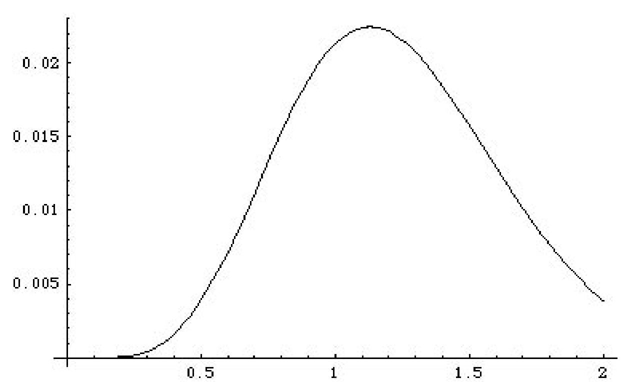
$$S(\lambda) = \frac{\partial \ln(\ell(\lambda \,|\, x_1, \cdots, x_n))}{\partial \lambda} = \frac{\partial\, n \ln \lambda + (\lambda - 1) \sum_{i=1}^n \ln x_i - \sum_{i=1}^n x_i^\lambda}{\partial \lambda}$$

$$= \frac{n}{\lambda} + \sum_{i=1}^n \ln x_i - \sum_{i=1}^n x_i^\lambda \ln x_i$$

(d) What is the second derivative of the log-likelihood function?

$$\frac{\partial S(\lambda)}{\partial \lambda} = -\frac{n}{\lambda^2} - \sum_{i=1}^n x_i^\lambda (\ln x_i)^2$$

(e) Suppose 5 values, 0.10, 0.25, 0.5, 1, and 2 are observed. Plot the resulting log-likelhood function



MLE is approximate $\widehat{\lambda} \simeq 1.13$

(f) What is the approximate sample variance?

$$\mathrm{Var}(\,\widehat{\lambda}\,) \simeq -\left(-\frac{5}{1.13^2} - \sum_{i=1}^{5} x_i^{1.13} \,(\ln\, x_i)^2 \right)^{-1} = (\,5.98\,)^{-1} = 0.167$$

(g) What is an approximate 95% confidence interval for λ?

Approach one (normal approximation)

$$1.13 \pm 1.96\sqrt{0.167} = 1.13 \pm 0.800 = (0.329, 1.459)$$

Approach two: values of λ that give a likelihood function value of 1/10 (or greater) of that under the MLE. This gives (0.433, 2.14)

(h) What is the p value for a likelihood ratio test that $\lambda = 0.5$? $\lambda = 1$?

$$2\log\left(\frac{\ell(1.13\,|\,x_1,\cdots,x_n)}{\ell(0.5\,|\,x_1,\cdots,x_n)}\right) = 2\log(5.82) = 3.52, \quad \Pr(\chi_i^2 > 3.52) = 0.06$$

$$2\log\left(\frac{\ell(1.13\,|\,x_1,\cdots,x_n)}{\ell(1\,|\,x_1,\cdots,x_n)}\right) = 2\log(1.055) = 0.106, \quad \Pr(\chi_i^2 > 0.106) = 0.745$$