Distributions of Functions of

Normal Random Variables

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The Unit (or Standard) Normal

The **unit** or **standard normal** random variable U is a normally distributed variable with mean zero and variance one, i. e. $U \sim \mathrm{N}(0,1)$. Note that if $x \sim \mathrm{N}(\mu, \sigma^2)$ that

$$\frac{x-\mu}{\sigma} \sim U \sim \mathrm{N}(0,1) \tag{1}$$

Thus to simulate a normal random variable with mean μ and variance σ^2, we can simply transform unit normals, as

$$x \sim \mu + \sigma U \sim \mathrm{N}(\mu, \sigma^2) \tag{2}$$

Consider n independent random variables $x_i \sim \mathrm{N}(\mu, \sigma^2)$, then $\overline{x} \sim \mathrm{N}(\mu, \sigma^2/n)$, and this

$$\frac{\overline{x}-\mu}{\sigma/\sqrt{n}} \sim U \sim \mathrm{N}(0,1) \tag{3}$$

Example 1. Let's construct a 95% confidence interval for the mean μ for Equation (3). First, let's use **`R`** to compute a value $U_{0.975}$ such that $\Pr(U \leq U_{0.975}) = 0.975$. In **`R`**, typing the command **`qnorm(0.975)`** returns **`1.96`**. Likewise, **`qnorm(0.025)`** returns **`-1.96`** and hence $\Pr(U \leq -1.96) = 0.025$. Hence,

$$\Pr(-1.96 \leq U \leq 1.96) = 0.95$$

Recalling Equation (3),

$$\Pr\left(-1.96 \leq U \leq 1.96\right) = \Pr\left(-1.96 \leq \frac{\overline{x}-\mu}{\sigma/\sqrt{n}} \leq 1.96\right)$$

Rearranging gives

$$\Pr\left(-1.96\sigma/\sqrt{n} \leq \overline{x}-\mu \leq 1.96\sigma/\sqrt{n}\right)$$

or

$$\Pr\left(-\overline{x} - 1.96\sigma/\sqrt{n} \leq -\mu \leq -\overline{x} + 1.96\sigma/\sqrt{n}\right)$$

which can also be written as

$$\Pr\left(\overline{x} - 1.96\sigma/\sqrt{n} \le \mu \le \overline{x} + 1.96\sigma/\sqrt{n}\right) = 0.95$$

giving a 95% confidence interval for the mean μ.

Central and Noncentral χ^2 Distributions

The χ^2 distribution arises from sums of squared, normally distributed, random variables — if $x_i \sim \mathrm{N}(0,1)$, then $u = \sum_{i=1}^{n} x_i^2 \sim \chi_n^2$, a **central** χ^2 distribution with n degrees of freedom. It follows that the sum of two χ^2 random variables is also χ^2 distributed, so that if $u \sim \chi_n^2$ and $v \sim \chi_m^2$, then

$$u + v \sim \chi^2_{(n+m)} \tag{4a}$$

Two other useful results are that if $x_i \sim \mathrm{N}(0, \sigma^2)$, then

$$\sum_{i=1}^{n} x_i^2 \sim \sigma^2 \cdot \chi_n^2 \tag{4b}$$

and for $\overline{x} = n^{-1} \sum_{i=1}^{n} x_i$,

$$\sum_{i=1}^{n} (x_i - \overline{x}\,)^2 \sim \sigma^2 \cdot \chi^2_{(n-1)} \tag{4c}$$

In this last case, subtraction of the mean causes the loss of one degree of freedom. Note that a special case of Equation (4c) is the sample estimate of the variance,

$$\mathrm{Var}(x) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x}\,)^2$$

so that

$$(n-1)\mathrm{Var}(x) \sim \sigma^2 \cdot \chi_n^2, \quad \text{implying} \qquad \frac{(n-1)\mathrm{Var}(x)}{\sigma^2} \sim \chi_n^2 \tag{4d}$$

Example 2. We can use Equation (4d) to construct a confidence interval on the true variance σ^2 given the sample variance $\mathrm{Var}(x)$, provided the x_i are drawn from independent normals with the same mean and variance σ^2.

First, recall that the **R** command **`qchisq(p,df)`** returns a value X such that $\Pr(\chi^2_{df} \leq X) = p$. Suppose sample size is $n = 20$. Since **`qchisq(0.975,19)`** returns a value of **`32.85`** and **`qchisq(0.025,19)`** returns **`8.91`**, we have

$$\Pr(8.91 \leq \chi^2_{19} \leq 32.85) = 0.95$$

From Equation 4d,

$$\Pr(8.91 \leq \chi^2_{19} \leq 32.85) = \Pr\left(8.91 \leq \frac{(n-1)\mathrm{Var}(x)}{\sigma^2} \leq 32.85\right)$$

Noting that for

$$\Pr(a \leq x \leq b) = \Pr\left(\frac{1}{a} \geq \frac{1}{x} \geq \frac{1}{b}\right)$$

we have

$$\Pr\left(8.91 \leq \frac{19\mathrm{Var}(x)}{\sigma^2} \leq 32.85\right) = \Pr\left(\frac{1}{8.91} \geq \frac{\sigma^2}{19\mathrm{Var}(x)} \geq \frac{1}{32.85}\right)$$

or

$$\Pr\left(\frac{19\mathrm{Var}}{8.91} \geq \sigma^2 \geq \frac{19\mathrm{Var}}{32.85}\right) = 0.95$$

or

$$\Pr\left(2.13\mathrm{Var} \geq \sigma^2 \geq 0.58\mathrm{Var}\right) = 0.95$$

giving the 95% confidence interval on the variance as 0.58Var to 2.13Var.

A **noncentral** χ^2 arises when the random variables being considered have nonzero means. In particular, if $x_i \sim \mathrm{N}(\mu_i, 1)$, then $u = \sum_{i=1}^n x_i^2$ follows a noncentral χ^2 distribution with n degrees of freedom and **noncentrality parameter**

$$\lambda = \sum_{i=1}^{n} \mu_i^2 \tag{5a}$$

and we write $u \sim \chi^2_{n,\lambda}$. As shown in Figure 1, increasing the noncentrality parameter λ shifts the distribution to the right. This is also seen by considering the mean and variance of u,

$$E(u) = n + \lambda \qquad \text{and} \qquad \sigma^2(u) = 2(n + 2\lambda) \tag{5b}$$

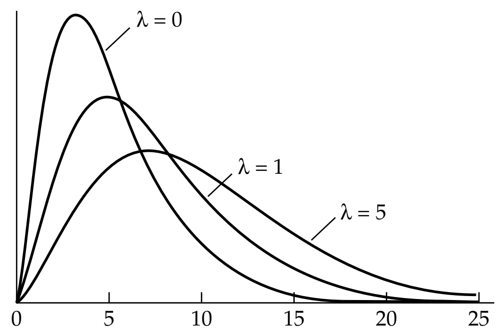

Figure 1 The probability distribution function for a noncentral χ^2. As the noncentrality parameter λ increases, the distribution is pulled to the right. We plot here a χ^2 random variable with n = 5 degrees of freedom and noncentrality parameters λ = 0 (a central χ^2), 1, and 5.

It follows directly from the definition that sums of noncentral χ^2 variables also follows a noncentral χ^2 distribution, so that if $u \sim \chi^2_{n,\,\lambda_1}$ and $v \sim \chi^2_{m,\,\lambda_2}$, then

$$(u+v) \sim \chi^2_{(n+m),(\lambda_1+\lambda_2)} \tag{5c}$$

Finally, Equations 4b,c can be generalized to noncentral χ^2 random variables as follows. Suppose $x_i \sim \mathrm{N}(\mu_i, \sigma^2)$, then

$$\sum_{i=1}^{n} x_i^2 \sim \sigma^2 \cdot \chi^2_{n,\lambda} \qquad \text{where} \qquad \lambda = \sum_{i=1}^{n} \frac{\mu_i^2}{\sigma^2} \tag{5d}$$

Turning the distribution of $\sum_{i=1}^{n} (\, x_i - \overline{x}\,)^2$, defining

$$\lambda^* = \sum_{i=1}^{n} \frac{(\mu_i - \overline{\mu}\,)^2}{\sigma^2}, \qquad \text{where} \qquad \overline{\mu} = \frac{1}{n}\sum_{i=1}^{n} \mu_i$$

then

$$\sum_{i=1}^{n} (x_i^2 - \overline{x}\,)^2 \sim \sigma^2 \cdot \chi^2_{n,\lambda^*} \tag{5e}$$

Note that if all the x_i have the same mean ($\mu_i = \mu = \overline{\mu}$), $\lambda^* = 0$ and the χ^2 is central, while if there is some variance in the means of the x_i, then distribution is a noncentral χ^2.

R provides commands for quantities of interest with noncentral χ^2 distributions.

- **`qchisq(p,df,ncp)`** returns a value X such that $\Pr(\chi^2_{df,ncp} \leq X) = p$

- **pchisq(X,df,ncp)** returns the probability that $\Pr(\chi^2_{df,ncp} \leq X)$
- leaving out the field for **ncp** returns these values for a central χ^2.

Student's t Distribution

If $x \sim \text{N}(\mu, \sigma^2)$, then for Equation 2, we have $(\overline{x} - \mu)/(\sigma/\sqrt{n}) \sim \text{N}(0,1)$, which allows for both hypothesis testing and construction of confidence intervals when σ^2 is known. When the variance is unknown, the above test statistic replaces the true (but unknown) variance σ^2 with the sample variance $\text{Var}(x)$,

$$t = \frac{\overline{x} - \mu}{\sqrt{\text{Var}/n}} \tag{6}$$

Notice that

$$t = \left(\frac{\overline{x} - \mu}{\sigma/\sqrt{n}}\right)\left(\frac{1}{\sqrt{\text{Var}/\sigma^2}}\right) = \frac{U}{\sqrt{\chi^2_{n-1}/(n-1)}}$$

Thus, we define a t **distributed random variable with** ν **degrees of freedom** by

$$t_\nu = \frac{U}{\sqrt{\chi^2_\nu/(\nu)}} \tag{7a}$$

A t random variable has mean zero and variance

$$\sigma^2\,(t_\nu) = 1 + \frac{2}{\nu - 2} \quad \text{for} \quad \nu > 2 \tag{7b}$$

The coefficient of kurtosis is $k_4 = 6/(\nu - 4)$, implying that the t distribution has heavier tails than a normal.

The **noncentral Student's t distribution** is defined as follows: If $x \sim \text{N}(\mu_o, \sigma^2)$, but we assume the correct mean is μ, then

$$t_{\nu,\lambda} = \frac{\overline{x} - \mu}{\sqrt{\text{Var}/n}}$$

is distributed as a noncentral t random variable with $n-1$ degrees of freedom and noncentrality parameter $\lambda = (\mu - \mu_0)/\sigma$.

Central and Noncentral F Distributions

The ratio of two χ^2-distributed variables leads to the F distribution. In particular, if $u \sim \chi^2_n$ and $v \sim \chi^2_m$, then the ratio of these two χ^2 variables divided by their respective degrees of freedom follows a **central F distribution/** with **numerator** and **denominator degrees of freedom** n and m (respectively), i.e., $(u/n)/(v/m) \sim F_{n,m}$. Since

$$\lim_{m \to \infty} F_{n,m} \to \frac{\chi^2_n}{n}$$

the F distribution can be approximated by a χ^2_n when the denominator degrees of freedom is large.

R provides commands for quantities of interest for F distributions.

- **`qf(p,df1,df2)`** returns a value X such that $\Pr(F_{df1,df2} \leq X) = p$
- **`pf(X,df1,df2)`** returns the probability that $\Pr(F_{df1,df2} \leq X)$

The **noncentral F distribution** results when the numerator χ^2 variable is noncentral. If $u \sim \chi^2_{n,\lambda}$ and $v \sim \chi^2_m$, then $F = (u/n)/(v/m)$ follows a noncentral F distributed with noncentrality parameter λ, and we write $F \sim F_{n,m,\lambda}$. As with the noncentral χ^2, increasing λ shifts the distribution further to the right. Again, this is seen in the mean and variance, with

$$E(F) = \frac{m}{m-2}\left(1 + \frac{2\lambda}{n}\right) \tag{A5.16a}$$

$$\sigma^2(F) = 2\left(\frac{m}{n}\right)^2 \left[\frac{(n+m)^2 + (n+2\lambda)(m-2)}{(m-2)^2(m-4)}\right] \tag{A5.16b}$$

R provides commands for quantities of interest for noncentral F distributions.

- **`pf(X,df1,df2,ncp)`** returns the probability that $\Pr(F_{df1,df2,ncp} \leq X)$
- the obvious command **`qf(p,df1,df2,ncp)`** does not work, as the same value is returned for all values of **`ncp`**.